Revealing geometric phases in modular and weak values with a quantum eraser

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(Received 20 August 2015; revised manuscript received 9 February 2016; published 28 April 2016)

We present a procedure to completely determine the complex modular values of arbitrary observables of pre- and postselected ensembles, which works experimentally for all measurement strengths and all postselected states. This procedure allows us to discuss the physics of modular and weak values in interferometric experiments involving a qubit meter. We determine both the modulus and the argument of the modular value for any measurement strength in a single step, by simultaneously controlling the visibility and the phase in a quantum eraser interference experiment. Modular and weak values are closely related. Using entangled qubits for the probed and meter systems, we show that the phase of the modular and weak values has a topological origin. This phase is completely defined by the intrinsic physical properties of the probed system and its time evolution. The physical significance of this phase can thus be used to evaluate the quantumness of weak values.

DOI: 10.1103/PhysRevA.93.042124

I. INTRODUCTION

In 1988, Aharonov, Albert, and Vaidman (AAV) introduced the weak value of a quantum observable \( \hat{A} \) from an extension of the von Neumann measurement scheme [1]. They pointed out that the result of a measurement involving a weak coupling between a meter and the observable \( \hat{A} \) of a system with a preselected initial state \( |\psi_i\rangle \) and a postselected final state \( |\psi_f\rangle \) depends directly on the weak value,

\[
A_w = \frac{\langle \psi_f | \hat{A} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle},
\]

which is an unbounded complex number. In particular, they showed that the shift of the average detected position due to postselection is proportional to the real part of the weak value. Since for weak measurements in the absence of postselection, this shift is proportional to the average of the observable \( \langle \psi_i | A | \psi_i \rangle / \langle \psi_i | \psi_i \rangle \), a direct but bold physical interpretation of the weak value assumes it somehow represents the average of \( \hat{A} \) in the pre- and postselected ensemble. They also related the imaginary part of the weak value to the shift of the average impulsion. Besides the AAV approach, weak values may also appear using a meter strongly coupled to the observable \( \hat{A} \) [2–7]. In these instances, the effective weak interaction is achieved by selecting particular initial states of the meter system, so that the probability of actually measuring \( \hat{A} \) is low and the probed system is left unperturbed most of the time. Therefore, both methods transform the standard von Neumann procedure to a weak measurement with a high incertitude.

Weak values and weak measurements have proven useful in many fields of physics and chemistry [8–20]. Nevertheless, the proper physical interpretation of weak values remains highly debated. For example, on the one hand, weak values were used to develop a time-symmetrized approach to standard quantum theory, i.e., the two-state vector formalism [21], where they appear as purely quantum objects. On the other hand, a purely classical view of the occurrence of unbounded, real weak values was proposed recently [22] (which is criticizable though [23–25]). In their paper, the authors even suggest that complex weak values have a classical explanation as well because, in practice, only the real or the imaginary part of weak values has been measured directly so far [22].

In the present work, we uncover a physical interpretation of complex weak values in terms of their polar representation (modulus and argument), which provides evidence for their quantumness. We devise an interferometric procedure to measure and discuss complex weak values in their polar representation instead of the usually determined real or imaginary part. Our procedure relies essentially on a joint phase and visibility measurement in a quantum interferometer where the meter system acts as a quantum eraser. Using simple cases exploiting entangled qubits, we relate the argument of the weak value to topological phases defined completely by the probed system states involved in the weak measurement. Additionally, our procedure works in conditions where the usual weak-measurement procedure fails completely: (i) for arbitrary measurement strengths (i.e., including strong measurements) and (ii) for orthogonal and nearly orthogonal initial and final probe states. It proceeds by optimizing the interference phase to simultaneously measure the modulus and the argument of the weak value in a single step.

II. THEORETICAL PROCEDURE

A. General probe state

Formally, our procedure implements a quantum controlled evolution, in which an arbitrary quantum system \( |\psi_i\rangle \), i.e., the probe, interacts with a qubit meter via the quantum gate [Fig. 2(a)],

\[
\hat{U}_{\text{GATE}} = \hat{\Pi}_r \otimes \hat{I} + e^{i\delta} \hat{\Pi}_{-r} \otimes \hat{U}_A,
\]

where \( \hat{\Pi}_{\pm r} \) are projectors acting on the meter and \( \delta \) is a phase factor first supposed to be null. The transformation \( \hat{U}_A = e^{-i\delta \hat{A}} \) is expressed in terms of a time-independent Hermitian operator \( \hat{A} \) and an arbitrary coupling strength \( g \), defined by the

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integral $g = \int g(t)dt$ [26]. After the gate interaction, the spin observable $\delta_q$ of the meter is measured. According to the final meter state, the information about whether the transformation $\hat{U}_A$ was applied on the probe can be preserved or erased, completely or partially. Finally, a projective measurement of the probe system postselects the vector state $|\psi_f\rangle$.

The average $\sigma_{mq}$ of the meter observable for a given pre- and postselected subensemble of the probe system is then (see Appendix A for a detailed derivation)

$$
\sigma_{mq} = 2P_m \left( \frac{m \cdot \bar{q}}{1 + P_m \bar{r} \cdot \bar{m}} \right) ReA_m + \left[ (\bar{r} \times \bar{m}) \cdot \bar{q} \right] ImA_m - \left( \frac{m \cdot \bar{q}}{1 + P_m \bar{r} \cdot \bar{m}} \right)^2 |A_m|^2. \tag{3}
$$

In this expression, the normalized vectors $\bar{m}$, $\bar{q}$, and $\bar{r}$ are the directions on the meter Bloch sphere determining the initial $|m\rangle$ and final $|q\rangle$ meter states as well as the projector state $|r\rangle$ controlling the interaction, respectively. The direction of $\bar{q}$ was chosen orthogonal to $\bar{r}$ to select maximally interfering pathways through the meter measurement: then, the gate action appears as a superposition of having applied both $\hat{U}_A$ and $\hat{I}$, and all information about the gate action is lost (quantum eraser condition). The parameter $P_m$ characterizes the purity of the initial meter state, ranging from 1 for pure states to 0 for a maximally mixed state. $A_m$ is defined as the modular value of the probe observable $\hat{A}$ for the pre- and postselected subsystem [27],

$$
A_m = \frac{\langle \psi_f | e^{-i \bar{q} \cdot \bar{A}} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle}. \tag{4}
$$

It appears from the action of projectors in (2). Modular values were not often reported as such in the literature because they are directly related to weak values in the usual weak approximation limit for small-coupling strengths, through a first-order polynomial development in $g$: $A_m = 1 - igA_w + O(g^2)$. Nevertheless, they characterize all projective couplings between the probe and meter systems, where they generalize by $g$ and the phase of interferometric experiments. They typically describe quantum-gate-type interactions [27] and quantum interference weak values in a nonperturbative way. They usually describe quantum-gate-type interactions [28–30], but also appear in photon trajectory experiments [17] for example. In the following, we relate the physical interpretation of modular values to the visibility and the phase of interferometric experiments.

In our procedure, the interaction strength is not determined by $g$. Instead, it reflects the probability of the application of $\hat{U}_A$ by the quantum gate, which is controlled by the measurement strength $\theta = \arccos (\bar{m} \cdot \bar{r})$, with $\theta \in [0, \pi]$. For a given vector $\bar{r}$ controlling the quantum gate-action, the quantum eraser condition $\bar{r} \cdot \bar{q} = 0$ constrains $\bar{q}$ to the red plane in Fig. 1(a). We choose particular final vectors $\bar{q}$ of the meter system in relationship to the initial vector $\bar{m}$ (characterizing the meter initial state) in order to determine the real and imaginary parts of the modular value from the average meter observable $\sigma_{mq}$. We pick the real part of $A_m$ when the three vectors $\bar{m}$, $\bar{r}$, $\bar{q}$ are coplanar $[\bar{q}_{Re} \perp \text{blue plane}]$ in Fig. 1(b)], so that $(\bar{r} \times \bar{m}) \cdot \bar{q} = 0$ in Eq. (3). We isolate the imaginary part with orthogonal initial and final states of the meter $[\bar{q}_{Im} \perp \text{blue plane}]$ in Fig. 1(c)], so that $\bar{m} \cdot \bar{q} = 0$ in Eq. (3).

For small measurement strengths $\theta \approx 0$ when the purity $P_m$ is close to one, we obtain the modular value according to the standard approximations of weak measurements,

$$
ReA_m \approx \frac{1}{\theta} \sigma_{mq}, \quad ImA_m \approx \frac{1}{\theta} \sigma_{mq}^* \tag{5}
$$

where the weak-measurement approximation effectively removes the nonlinear dependence of Eq. (3) on the modulus of the modular value (see denominator).

For an arbitrary measurement strength, we seek instead to measure the modular value in its polar form to directly assess its modulus $|A_m|$ and argument $\varphi = \arg A_m$. We introduce an additional unitary transformation $\hat{R}_\xi$ in the meter path. It creates a relative phase shift $\xi$ between the orthogonal states $|r\rangle$ and $|-r\rangle$ that is effectively equivalent to a rotation of the modular value in the complex plane. When the phase shift precisely compensates the argument of the modular value (i.e., when $\xi = \varphi$), this rotation aligns the modular value with the real axis. Choosing the meter configuration $\bar{q}_{Re}$ that picks the real part of the modular value now provides its full modulus, while its argument is equal to the introduced phase shift. In practice, our procedure implements a quantum interferometer exploiting entanglement to measure the two quantities concurrently. Indeed, the expression for the joint probability outcome $P_{\text{joint}}$ of the meter and the probe measurements is proportional to

$$
P_{\text{joint}} \propto 1 + V \cos(\varphi - \xi), \tag{6}
$$

where $\xi$ is a phase shift introduced by the interaction of the meter with the probe, and $V$ is the visibility of the interference phenomenon, where $V$ is the visibility and $\varphi - \xi$ represents the phase. Experimentally, the visibility is determined by measuring the maximum and the minimum of the joint probability, denoted by $P_{\max}$ and $P_{\min}$, respectively.

$$
V = \frac{P_{\max} - P_{\min}}{P_{\max} + P_{\min}}. \tag{7}
$$

When the phase introduced by $\hat{R}_\xi$ equals the argument of the modular value, the maximum of the joint probability is obtained for the meter vector $\bar{q}_{Re}$, while its minimum is obtained for the orthogonal state $-\bar{q}_{Re}$. The two situations correspond to constructive and destructive interferences in the joint measurement, respectively. The visibility depends on the coupling strength and the modulus of the modular value,

$$
V = \frac{2P_m \tan \left( \frac{\xi}{2} \right)}{C_{\theta + \pi} + C_\theta \tan^2 \left( \frac{\xi}{2} \right)} |A_m|^2, \tag{8}
$$

FIG. 1. Representation in the Bloch sphere of the relevant (a)–(c) meter states. (a) The red plane is perpendicular to the control state $\bar{r}$. It contains all final meter states $\bar{q}$ and $-\bar{q}$ implementing the quantum eraser condition. (b),(c) The blue plane contains the initial meter state $\bar{m}$ and the control state $\bar{r}$. The final meter states (b) $\bar{q}_{Re}$ in the blue plane and (c) $\bar{q}_{Im}$ perpendicular to it measure the real and imaginary parts of the modular value, respectively.
with coefficients $C_p$ defined by

$$C_p = \frac{1 + P_m}{2} + \frac{1 - P_m}{2} \cot^2 \frac{\varepsilon}{2}.$$  

(9)

This quadratic equation provides two solutions for the modulus of the modular value:

$$|A_m|_2 = \frac{1 \pm \sqrt{1 - C_{\theta} P_m^2 V^2}}{C_{\theta} \tan \left( \frac{\theta}{2} \right)} P_m.$$  

(10)

The solution $|A_m|_2$ corresponds to $|A_m|$, if the condition

$$\tan^2 \left( \frac{\theta}{2} \right) C_{\theta} |A_m|^2 \leq 1$$  

(11)

is verified, and $|A_m|_2 = |A_m|$ otherwise. Together, they provide the characterization of the modulus of the modular value for an arbitrary coupling strength. It is directly related to the visibility. In particular, the weak-measurement approximation gives $|A_m| \approx V/\theta$, similarly to Eq. (5). In this expression of the modular value, the visibility plays the same role as the pointer shift in weak values. This shows a strong connection between modular values and weak interferometric experiments.

### B. Qubit probe state

Now we consider the connection between modular and weak values to gain insight into the physics of weak values. The previously arbitrary probe system becomes a qubit and the probe transformation $\hat{U}_A = e^{-i \hat{G}}$ is a rotation operator involving the Pauli observable $\sigma_n = \vec{n} \cdot \vec{\sigma}$ (with $\vec{n}$ a unit vector). We set a strong AAV coupling strength $g = \pi$. Then, $\hat{U}_A = -i\sigma_n$ and the quantum gate acting on the two qubits becomes

$$\hat{U}_{\text{GATE}} = \hat{\Pi}_i \otimes \hat{I} + \hat{\Pi}_{i-f} \otimes \sigma_n.$$  

(12)

where the phase factor $\delta$ in (2) was set to $\frac{\pi}{2}$. This shows the equivalence of modular and weak values of $\sigma_n$ (also see [27]). We can thus apply our scheme to determine an arbitrary weak value of the Pauli operator in its polar representation. Interestingly, the argument of the weak value depends only on the evolution from its initial to final state as defined by the operator $\hat{U}_A$ (a related result is mentioned in [31]). This phase has a topological component, similar to the Pancharatnam geometric phase. Its value is proportional to the solid angle $\Omega_{\text{nif}}$ delimited by four vectors on the Bloch sphere [see Fig. 2(b) and derivation in Appendix B for details]:

$$\arg \left( \frac{\langle \psi_f | \sigma_n | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} \right) = \arctan \left( \frac{\vec{n} \times \vec{i} \cdot \vec{f}}{\vec{n} \cdot \vec{i} + \vec{n} \cdot \vec{f}} \right) = -\frac{1}{2} \Omega_{\text{nif}}.$$  

(13)

This geometric phase is completely defined by intrinsic properties of the probe system. It is observed directly in the interferometric experiment but does not depend on the meter properties, contrary to the pointer shift in usual weak measurement (which depends on $g$). It emphasizes the quantum origin of the argument of the complex weak value and its relationship to physical properties of the probe system.

### III. EXPERIMENT

Experimentally, we implement a conceptual CNOT gate, $\hat{U}_{\text{GATE}} = \hat{\Pi}_{0} \otimes \hat{I} + \hat{\Pi}_{1} \otimes \sigma_x$. The initial meter state $\rho_m = \frac{1}{2} (|0\rangle \langle 0| + |1\rangle \langle 1|)$ is verified, and

$$|A_m|_2$$

(10)

is otherwise. Together, they provide the characterization of the modulus of the modular value for an arbitrary coupling strength. It is directly related to the visibility. In particular, the weak-measurement approximation gives $|A_m| \approx V/\theta$, similarly to Eq. (5). In this expression of the modular value, the visibility plays the same role as the pointer shift in weak values. This shows a strong connection between modular values and weak interferometric experiments.

- **FIG. 2. Quantum controlled evolution:** (a) protocol, (b) representation in the Bloch sphere of the relevant probe states, and (c) experimental setup. (b) Probe Bloch sphere with initial $i$ and final $f$ states, $\sigma_n$ observable rotation axis $\vec{n}$, and the mirror image of $\vec{n}'$ with respect to the $\vec{n}$ axis. The solid angle $\Omega_{\text{nif}}$ associated to the geometric phase in Eq. (13) is obtained by conversely following the three great circle arcs $i \rightarrow n \rightarrow i'$ (blue), $i' \rightarrow f$ (red), and $f \rightarrow i$ (red). (c) The setup comprises three areas: the state preparation with the two qubit generation (I), the meter measurement by detectors $D_1$ and $D_2$ (II), and the final probe postselection by $D_3$ (III). The coincidence counts $N_{12}$ and $N_{23}$ are acquired by four single-photon counting modules (SPCMs) placed in the meter and probe paths.

We postselect the probe polarization $|\psi_f\rangle = \cos (\alpha) |H\rangle + \sin (\alpha) |V\rangle$ at detector $D_2$. Detectors $D_1$ and $D_2$ measure the meter polarization (diagonal $|D\rangle$ and antidiagonal $|A\rangle$ states, respectively). We adjust the phase $\xi$ by tilting a birefringent $Z$-cut quartz plate in the meter path to obtain the interference visibility $V$ from the coincidence counts. When coincidence counts $N_{13}^{\text{max}}$ are maximal for detectors $D_1$ and $D_3$ (constructive interference), coincidence counts $N_{13}^{\text{min}}$ for detectors $D_2$ and $D_3$ are minimal (destructive interference). Then, the phase $\xi$ equals the argument $\varphi$ of the weak value
The solid angle or $\pi_0$ positive or negative solution larger or smaller than unity (violet solid horizontal line) admit the $(black triangles)$. (c) Positive or negative solution criterion. Values are (a),(b)

$P_{|\psi_f\rangle}$ fitting the theoretical visibility to the data (r).

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IV. RESULTS AND DISCUSSION

Figure 3 presents the visibility and the phase for three different initial meter states inconciliable with weak-measurement approximations. The corresponding strengths $\theta$ were determined from the density operator of the biphoton using quantum state tomography [33]. The purities were rather estimated by minimization method because $V$ is highly sensitive to $P_{m}$. The latter step may be skipped if the meter is supposed in a pure initial state (for details, see Appendix E), as is usually done in the literature. We chose pre- and postselected states for which the argument of $\sigma_{x,w}(\alpha)$ [Fig. 3(b)] takes only the two values 0 or $\pi$ for simplicity. It determines the sign of the weak value. The solid angle $\Omega_{\text{ini/f}}$ related to the geometric phase (13) is defined in the $OZX$ plane of the Bloch sphere [see Fig. 3(b)]: the great circle arcs $i \rightarrow n \rightarrow i'$ make a full circle or compensate each other depending on the postselected state $f(\alpha)$. The visibility [Fig. 3(a)] provides the modulus $|\sigma_{x,w}|$, using the two solutions obtained in relation (10). The switch between them occurs at the maximum of the visibility. The criterion (11) determining this switch is measured from the coincidence count ratio $N_{22}^c/N_{13}^c$ with horizontal $|H\rangle$ (detector $D_1$) and vertical $|V\rangle$ (detector $D_2$) meter polarizations. In this case, $\vec{r} = \vec{q}$ and the meter measurement completely reveals the probe state after the quantum-gate interaction (no information erasure). The theoretical switch angle and the measured criterion agree strongly except for strengths approaching the range of weak measurements ($\theta_f$), where a difference of 2–4° is observed due to increasing experimental noise [Fig. 3(c)].

The full weak values determined using a strong ($\theta_2$) or a weaker ($\theta_3$) strength are compared in Fig. 4(a). Both setups

FIG. 3. (a) Visibility and (b) argument as a function of the postselected polarization $|\psi_f\rangle = \cos (\alpha) |H\rangle + \sin (\alpha) |V\rangle$ with preselected $|H\rangle$ polarization for three initial meter states: $\theta_1 = 0.499 \pi$ and $P_{m} = 0.882 \pm 0.002$ (red squares), $\theta_2 = 0.297 \pi$ and $P_{m} = 0.836 \pm 0.002$ (blue circles), $\theta_3 = 0.092 \pi$ and $P_{m} = 0.956 \pm 0.001$ (black triangles). (c) Positive or negative solution criterion. Values larger or smaller than unity (violet solid horizontal line) admit the positive or negative solution $|\sigma_{x,w}|$, respectively. Final meter states are (a),(b) $|D\rangle$ and $|A\rangle$ and (c) $|H\rangle$ and $|V\rangle$. Gray solid lines represent theoretical curves.

$\sigma_{x,w} = \langle \psi_f(\alpha)|\hat{\sigma}_x|H\rangle/\langle \psi_f(\alpha)|H\rangle$, while the visibility is given by Eq. (7): $V = \frac{N_{22}^c - N_{13}^c}{N_{22}^c + N_{13}^c}$.

This scheme improves the signal-to-noise ratio (SNR) of a weak measurement. The fundamental reason behind this improvement is that we measure a larger signal, which is thus less sensitive to noise. Indeed, the signal is proportional to the whole modulus of the weak value, which is larger than the usually measured real (or imaginary) part of the weak value. A complete demonstration is shown in Appendix D.

FIG. 4. Weak values determined from phase and visibility measurement for the three strengths $\theta_1$ (red squares), $\theta_2$ (blue circles), and $\theta_3$ (black triangles), and from the standard weak-measurement technique (violet diamonds) using relation (5) with the weak strength $\theta_f = 0.025 \pi$ and purity $P_{m} = 0.982 \pm 0.001$. All setups use a $|D\rangle,|A\rangle$ basis for the final meter measurement (the additional measurement in the $|L\rangle,|R\rangle$ meter basis required by standard weak measurements was not performed since Im$\sigma_{x,w}(\alpha) = 0$ here).
provide excellent agreement with the theoretical curve, except at the solution switch where the accuracy of the setup using weaker-measurement strengths decreases [see insets (a.1) and (a.2) of Fig. 4(a)]. In Fig. 4(b), we compare our method to the standard weak-measurement technique. For a small modulus of the weak value $\sigma_{r,w}(\alpha)$, both techniques provide results that are close to theoretical predictions. However, for large moduli, the weak-measurement approximation breaks down completely [see zoom (b.1) of Fig. 4(b)] for a wide range of postselected states approaching orthogonality to the preselected state. Weak-measurement results are useless there and only our method works.

V. CONCLUSIONS

In conclusion, the presented quantum eraser procedure exploits a qubit meter to directly measure the modulus and the argument of complex modular and weak values for arbitrary measurement strengths. The connection between modular and weak values allowed us to directly investigate weak values of qubit systems in their polar representation by performing a one-step visibility and phase measurement. In this case, the argument of the weak value is associated to a quantum geometric phase, which has a noncontroversial physical meaning. This direct relevance of the polar form of the weak value to the intrinsic physical properties of the system evolution shows that the interpretation of past and present experiments involving weak values ought not be limited to the consideration of their real and imaginary parts. Our method to determine weak values requires fewer measurements and does not suffer the limitations of the standard weak-measurement technique for large weak values, while it is applicable for both weak- and strong-measurement conditions. Experimentally, this opens the way to exploiting with greater accuracy the measure of weak values, particularly for nearly orthogonal pre- and postselected states.

ACKNOWLEDGMENTS

Y.C. is a research associate of the Belgian Fund for Scientific Research F.R.S.-FNRS. B.K. acknowledges financial support from the Action de Recherche Concertée (BIOSTRUCT project) of the University of Namur (UNamur) and the support from the COST(EU)-Nanoscale Quantum Optics (MP1403) action. The authors would like to thank Professor B. Hespel and Professor P. A. Thiry for fruitful discussions and support, without which this work would not have been possible. We also thank S. Mouchet for the careful reading of the paper and the insightful remarks, as well as J.-P. van Roy for the upgrade and the automation of the whole photon-detection system.

APPENDIX A: METER AVERAGE

During the quantum controlled evolution, the preselected probe state $\hat{\rho}_i = |\psi_i\rangle\langle\psi_i|$ interacts with the qubit meter $\hat{\rho}_m = \frac{1}{2}(I + P_m \hat{m} \cdot \hat{\sigma})$ via the quantum gate,

$$\hat{U}_{\text{GATE}} = \hat{U}_r \otimes I + e^{i\phi} \hat{U}_{-r} \otimes \hat{U}_A,$$

(A1)

with the meter projectors $\hat{1}_{\pm r} = \frac{1}{2}(I \pm \hat{m} \cdot \hat{\sigma})$. The vectors $\hat{m}$ and $\hat{\sigma}$ are normalized three-dimensional vectors pointing on the Bloch sphere. $P_m$ is the purity of the initial meter state. After the gate interaction, the whole system (meter and probe) state is

$$\hat{\rho} = \hat{U}_r \hat{\rho}_m \hat{U}_r^\dagger \otimes \hat{\rho}_i + \hat{U}_{-r} \hat{\rho}_m \hat{U}_{-r}^\dagger \otimes \hat{U}_A \hat{\rho}_i \hat{U}_A^\dagger$$

$$+ \hat{U}_r \hat{\rho}_m \hat{U}_r^\dagger \otimes \hat{U}_A \hat{\rho}_i + \hat{U}_{-r} \hat{\rho}_m \hat{U}_{-r}^\dagger \otimes \hat{U}_A \hat{\rho}_i \hat{U}_A^\dagger.$$  

(A2)

If we express the initial meter state, as well as the projectors, using their associated vectors on the Bloch sphere, the density operator $\hat{\rho}$ can be transformed to

$$\hat{\rho} = \frac{1}{2}(1 + P_m \hat{m} \cdot \hat{\sigma})(1 + \hat{m} \cdot \hat{\sigma})^{-1} \hat{\rho}_i$$

$$+ (1 - P_m \hat{m} \cdot \hat{\sigma})(1 - \hat{m} \cdot \hat{\sigma}) \otimes \hat{U}_A \hat{\rho}_i \hat{U}_A^\dagger$$

$$+ P_m [\hat{m} \cdot \hat{\sigma} - (\hat{m} \cdot \hat{\sigma})(\hat{m} \cdot \hat{\sigma})] \otimes \hat{U}_A \hat{\rho}_i$$

$$- i(\hat{m} \times \hat{\sigma}) \otimes \hat{U}_A \hat{\rho}_i$$

$$+ P_m [\hat{m} \cdot \hat{\sigma} - (\hat{m} \cdot \hat{\sigma})(\hat{m} \cdot \hat{\sigma})] \otimes \hat{U}_A \hat{\rho}_i$$

$$+ i(\hat{m} \times \hat{\sigma}) \otimes \hat{U}_A \hat{\rho}_i].$$  

(A3)

According to our quantum measurement scheme, the spin observable $\hat{d}_q$ of the meter is then measured and the probe is postselected by the state $|\psi_f\rangle$. Since the spin operator $\hat{d}_q$ verifies $\hat{d}_q = \hat{U}_{-q} \hat{d}_q \hat{U}_{+q}$, we first have to calculate the joint probabilities $P^+_{\text{joint}}$ and $P^-_{\text{joint}}$ of postselecting the probe state $|\psi_f\rangle$ while measuring the meter observable $\hat{d}_q$ or measuring the meter observable $\hat{d}_{-q}$, respectively:

$$P^\pm_{\text{joint}} = \text{tr}((|\psi_f\rangle\langle\psi_f| \otimes \hat{d}_{\pm q})\hat{\rho}),$$

where $\hat{d}_{\pm q} = \frac{1}{2}(I \pm \hat{r} \cdot \hat{\sigma}).$ We find then,

$$P^\pm_{\text{joint}} = \frac{1}{2}(1 + P_m \hat{m} \cdot \hat{\sigma})(1 \mp \hat{r} \cdot \hat{\sigma}) \langle|\psi_f\rangle |\psi_f\rangle|^2$$

$$+ (1 - P_m \hat{m} \cdot \hat{\sigma})(1 \mp \hat{r} \cdot \hat{\sigma}) \langle|\psi_f\rangle |\hat{U}_A |\psi_f\rangle|^2$$

$$\pm 2P_m [\hat{m} \cdot \hat{\sigma} - (\hat{r} \cdot \hat{\sigma})(\hat{r} \cdot \hat{\sigma})] \times $$

$$\text{Re}(|\psi_f\rangle\langle\hat{U}_A |\psi_f\rangle |\psi_f\rangle |\hat{U}_A |\psi_f\rangle),$$

(A4)

By applying the quantum eraser condition $\hat{r} \cdot \hat{q} = 0$, and by considering

$$\sigma^m_q = \frac{P^+_{\text{joint}} - P^-_{\text{joint}}}{P^+_{\text{joint}} + P^-_{\text{joint}}}.$$  

(A6)

it can be finally shown that

$$\sigma^m_q = 2P_m \langle\hat{m} \cdot \hat{q}\rangle \text{Re}A_m + \langle[\hat{r} \times \hat{m}] \cdot \hat{q}\rangle \text{Im}A_m$$

$$- (1 + P_m \hat{r} \cdot \hat{m}) + (1 - P_m \hat{r} \cdot \hat{m}) |A_m|^2.$$  

(A7)

APPENDIX B: TOPOLOGICAL COMPONENT OF THE WEAK-VALUE ARGUMENT

As described in our work, the argument of the weak value of the spin operator $\hat{d}_q$ verifies the two equalities:

$$\arg \left( \frac{f(\hat{d}_q|i\rangle)}{f|i\rangle} \right) = \arctan (\frac{n \times f}{n \hat{i} + n \hat{f}}) = -\frac{1}{2} \Omega_{m'f}.$$  

(B1)
The first equality results immediately from the definition of the argument of the weak value, considering the following expression:

\[
\arg \left( \frac{\langle f | \hat{\sigma}_n | i \rangle}{\langle f | i \rangle} \right) = \arctan \left( \frac{\text{Im} \left( \langle f | \hat{\sigma}_n | i \rangle \langle i | f \rangle \right)}{\text{Re} \left( \langle f | \hat{\sigma}_n | i \rangle \langle i | f \rangle \right)} \right),
\]

(B2)
in which \(\langle f | \hat{\sigma}_n | i \rangle \langle i | f \rangle\) can be expressed as a function of directions on the Bloch sphere,

\[
\langle f | \hat{\sigma}_n | i \rangle \langle i | f \rangle = \frac{1}{2} [\hat{n} \cdot \hat{f} + \hat{n} \cdot \hat{j} + j(\hat{n} \times \hat{i}) \cdot \hat{f}],
\]

(B3)

where \(j\) is the imaginary unit, \(\hat{i}\) and \(\hat{j}\) describe the pure initial and final qubit states, respectively, and \(\hat{n}\) gives the rotation axis associated to the observable \(\hat{\sigma}_n\). For the second equality in (B1), our approach is inspired by the work of Martinez et al. [3], where they studied the geometric characteristics (amplitude and phase) of polarization modulation optical devices on the Poincaré sphere.

The trajectory of a pure qubit state on the Bloch sphere corresponding to the unitary transformation \(\hat{\sigma}_n\) is a nongeodesic opened arc. Consequently, the resulting state \(|f\rangle = \hat{\sigma}_n |i\rangle\) is no longer in phase with the initial state \(|i\rangle\). To yield nonetheless an expression in terms of a solid angle for the accumulated phase, let us express the resulting state as \(|f\rangle = e^{i\varphi_{i\rightarrow f}} |i\rangle\), where the phase \(\varphi_{i\rightarrow f}\) is due to the nongeodesic movement of \(|i\rangle\) to the output state \(|f\rangle\). The vector \(\hat{i}'\) is entirely defined by \(\hat{i}' = 2(\hat{n} \cdot \hat{i})\hat{n} - \hat{i}\) (essentially, \(\hat{i}'\) is the mirror image of \(\hat{i}\) with respect to the \(\hat{n}\) axis). The additional phase \(\varphi_{i\rightarrow f}\) is determined by projecting \(|f\rangle\) onto the orthogonal eigenvectors \(|n\rangle\) and \((-n)\rangle\) of the operator \(\hat{n}\). By considering

\[
\hat{n} = |n\rangle \langle n| - |-n\rangle \langle -n|,
\]

we conclude that the projection of \(|f\rangle\) onto the eigenvector \(|n\rangle\) yields the following two relations:

\[
\langle n | f \rangle = e^{i\varphi_{i\rightarrow f}} \langle n | i \rangle = \langle n | i \rangle.
\]

(B5)
The moduli \(|\langle n | i \rangle| = |\langle n | i' \rangle|\) are equal since \(\hat{i}\) and \(\hat{i}'\) are mirror images with respect to \(\hat{n}\). Consequently, only the accumulated total phase of the opened loops, known as the Pancharatnam connection, remain in (B5),

\[
\varphi_{i\rightarrow f} = \varphi_{i\rightarrow i'} - \varphi_{n\rightarrow i'} = \arg \langle n | i \rangle - \arg \langle n | i' \rangle.
\]

(B6)

In practice, the Pancharatnam connection argument \(\arg \langle b | a \rangle\) relating arbitrary states \(|a\rangle\) and \(|b\rangle\) is calculated by determining the spherical quadrangle \(\Omega_{a_b b_a a_b}\), in the Bloch sphere (Fig. 5), where the supplemental vertices \(|a\rangle\) and \(|b\rangle\) are well-defined vectors. To understand how they are determined, we must express their position in the spherical coordinate system. By convention, \(2\eta\) corresponds to the azimuth angle and \(2\chi\) to the polar angle. In this representation, a pure state on the Bloch sphere is defined by

\[
\vec{a}(\chi, \eta) = \begin{pmatrix} \cos(2\eta) \cos(2\chi) \\ \sin(2\eta) \cos(2\chi) \\ \sin(2\chi) \end{pmatrix},
\]

(B7)

and the Pancharatnam connection is given by

\[
\arg \langle b | a \rangle = \arctan \left( \tan(\eta_a - \eta_b) \frac{\sin(\chi_a + \chi_b)}{\cos(\chi_a - \chi_b)} \right).
\]

(B8)

The connection is in phase, i.e., \(\arg \langle b | a \rangle = 0\), for transports with the same azimuth angles \(\eta\) and for transformations happening around the equator of the Bloch sphere, i.e., for the polar angle \(\chi = 0\). These two kinds of transports are known as horizontal lifts along the geodesic connecting the states \(|a\rangle\) and \(|b\rangle\) on the Bloch sphere. The states \(|a\rangle = |\eta_a, 0\rangle\) and \(|b\rangle = |\eta_b, 0\rangle\) are fixed with the same azimuth angle as \(|a\rangle\) and \(|b\rangle\), respectively, and with a polar angle \(\chi = 0\). The closed loop \(|a\rangle \rightarrow |b\rangle \rightarrow |b\rangle \rightarrow |a\rangle \rightarrow |a\rangle\) along the geodesic arcs determines the spherical quadrangle \(\Omega_{a_b b_a a_b}\), which is equivalent to \(\arg \langle b | a \rangle\):

\[
\arg \langle b | a \rangle = \arg \langle a | a_b \rangle + \arg \langle a_b | b_a \rangle + \arg \langle b_a | b \rangle + \arg \langle b | a \rangle = -\Omega_{a_b b_a a_b}. \quad (B9)
\]

Note that the sign present in front of the solid angle for a given sequence of states is positive when the sequence is followed counterclockwise and is negative when the sequence is followed clockwise. The sign of the solid angle changes when the sequence of projections is inverted, \(\Omega_{a \rightarrow b \rightarrow c} = -\Omega_{b \rightarrow c \rightarrow a}\). It is possible to express a solid angle linking three vertices as a sum of three spherical quadrangles [3],

\[
\Omega_{abc} = \Omega_{a_b b_a a_b} + \Omega_{b_c c_b b_c} + \Omega_{c_a a_c c_a}, \quad (B10)
\]

where each spherical quadrangle contains two vertices of the initial solid angle.

We use the decomposition property of Eq. (B10) to rewrite the expression giving \(\varphi_{i\rightarrow i'}\) (B6) according to

\[
\varphi_{i\rightarrow i'} = -\frac{\Omega_{i\rightarrow i''} + \Omega_{i''\rightarrow i'}}{2}, \quad (B11)
\]

where we made use of Eq. (B9) to express the connections appearing in (B6). Following the indices may be tedious but, essentially, Eqs. (B6) and (B9) show together that the expression of the phase \(\varphi_{i\rightarrow i'}\) includes a sum of two spherical quadrangles; then we used Eq. (B10) to express the sum of these two spherical quadrangles as a function of the third spherical quadrangle and of the spherical triangle appearing in Eq. (B10).
Furthermore, expression (B11) points out that the non-geodesic phase \( \phi_{i \rightarrow i'} \) is the sum of the geometric phase of the closed loop \( |i\rangle \rightarrow |n\rangle \rightarrow |i'\rangle \rightarrow |i\rangle \) (first term) and the phase of the Pancharatnam connection \( |i\rangle \rightarrow |i'\rangle \) (second term).

Using the last results, the argument of the weak value of \( \hat{\sigma}_n \) is

\[
\arg \left( \frac{\langle f | \hat{\sigma}_n | i \rangle}{\langle f | i \rangle} \right) = \arg \left( e^{i \phi_{i \rightarrow i'}} \frac{\langle f | i' \rangle}{\langle f | i \rangle} \right) \\
= \arg \left( e^{i \phi_{i \rightarrow i'}} - \frac{\Omega_{ii'} f - \Omega_{i'i'} f^*}{2} \right) \\
= \frac{-\Omega_{ii''} f^*}{2} - \frac{\Omega_{i'i''} f}{2} \\
= \frac{-\Omega_{ii''} f}{2} - \frac{\Omega_{i'i''} f^*}{2} \\
= \frac{-\Omega_{ii''} f}{2}.
\]

Equality (a) results from the definition of the states \( |i'\rangle \) and \( |i''\rangle \). Equality (b) expresses the Pancharatnam connections in terms of solid angles using Eq. (B9). Equality (c) takes the argument of the previous expression. Equality (d) exploits the decomposition property of Eq. (B10). Equality (e) follows from the expression of \( \phi_{i \rightarrow i'} \) in (B11). Equality (f) is due to canceling terms. Equality (g) combines the two spherical triangles in one spherical quadrangle (as the paths \( i \rightarrow i' \) and \( i' \rightarrow i \) present in the triangles cancel each other).

**APPENDIX C: EXPERIMENTAL EQUIPMENT**

Experimentally, we implement the polarization-entangled biphoton state after the conceptual CNOT gate using two orthogonal nonlinear BBO crystals in the “sandwich configuration” [32]. Each crystal is 1 mm thick and cut for type-I phase matching with \( \theta = 29.2^\circ \) and \( \varphi = 90^\circ \). Via the nonlinear interaction of spontaneous parametric down conversion (SPDC), the pump laser (blue diode DL-7146-101S from SANYO Electric Co.) centered at 407 nm generates two polarization-entangled photons at 814 nm. The laser diode is controlled by temperature (Thorlabs TED 200C) and current (Thorlabs LDC202C) controllers. It produces a continuous laser output power of 60 mW. We use four single-photon counting modules (SPCM-AQ4C from Perkin-Elmer) for the joint polarization measurement of the meter or probe photons. The polarization basis is selected by half- and quarter-wave plates followed by a polarizing beam splitter (RCHP-15.0-CA-670-1064 from CVI Melles Griot). Before detection, the photons are coupled into multimode fibers and filtered using low-pass filters (FGL780 from Thorlabs). About 4000 total coincidence counts per second are acquired by using a homemade FPGA (SE3BOARD from Xilinx) coincidence counter.

**APPENDIX D: SIGNAL-TO-NOISE RATIO**

By definition, the signal-to-noise ratio (SNR) is the ratio of the magnitude of the expected meter shift to the standard deviation [2].

For given pre- and postselected probe states, we consider a final meter outcome which follows a binomial distribution: the meter qubit is measured either by detector \( D_1 \) (and contributes to \( N_{13}^{\text{max}} \)) or by detector \( D_2 \) (and contributes to \( N_{13}^{\text{min}} \)). The expectation value of \( N_{13}^{\text{min}} \) is therefore \( E[N_{13}^{\text{min}}] = p(2|3)N \) and its variance \( \Delta N_{13}^{\text{min}} = p(2|3)(1-p(2|3))N \). \( N = N_{13}^{\text{max}} + N_{13}^{\text{min}} \) is the total number of pre- and postselected detector events and \( p(2|3) \) is the conditional probability to trigger the meter detector \( D_2 \) for a given probe detection by \( D_1 \). Consequently, the expectation value of the measured visibility is

\[
E_V = E \left[ \frac{N_{13}^{\text{max}} - N_{13}^{\text{min}}}{N_{13}^{\text{max}} + N_{13}^{\text{min}}} \right] \\
= 1 - \frac{2}{N} E[N_{13}^{\text{min}}] \\
= 1 - 2p(2|3) \\
= V,
\]

(D1)

where the last equality follows directly from the definition of the conditional probability: \( p(2|3) = \frac{P_{\text{max}}}{P_{\text{max}} + P_{\text{min}}} \), with \( P_{\text{max}} \) and \( P_{\text{min}} \) the maximum and the minimum of the joint probability of the measurement protocol. The corresponding variance is

\[
\Delta_V = \Delta \left( \frac{N_{13}^{\text{max}} - N_{13}^{\text{min}}}{N_{13}^{\text{max}} + N_{13}^{\text{min}}} \right) \\
= \frac{4}{N^2} \Delta (N_{13}^{\text{min}}) \\
= \frac{4p(2|3)(1-p(2|3))}{N} \\
= \frac{1 - V^2}{N},
\]

(D2)

where we used the relationship \( p(2|3) = \frac{1 - \sqrt{V}}{2} \). This leads to the standard deviation,

\[
\sigma_V = \sqrt{\frac{1 - V^2}{N}}.
\]

(D3)

Finally, the signal-to-noise ratio of the presented measurement scheme is

\[
\text{SNR} = \frac{V}{\sqrt{1 - V^2}} \sqrt{N}.
\]

(D4)

The standard protocol determines the real (or the imaginary) part of the modular value by measuring the meter observable \( \hat{\sigma}_{q_n} \) (or \( \hat{\sigma}_{q_n} \)). In this case, the visibility \( V \) in the signal-to-noise relation (D4) is replaced by the absolute value of the meter average \( |\sigma_{q_n}| \) (or \( |\sigma_{q_n}| \)). In the weak-measurement limit, the latter is related to the modular value by the approximation \( \sigma_{q_n} \approx \theta |\text{Re} A_m| \) (or \( |\text{Re} A_m| \)). This expression is similar to the one obtained for our scheme in the weak-measurement limit, relating the visibility to the modulus of the modular value: \( V \approx \theta |A_m| \). Because the modulus of a complex number is always larger than or equal to its real and imaginary parts (\( |A_m| \geq \text{Re} A_m \) and \( |A_m| \geq \text{Im} A_m \)), our scheme improves the

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signal-to-noise ratio of the weak measurement compared to the standard protocol. (Additionally, it works when the weak-measurement approximation fails.)

APPENDIX E: PRELIMINARY METER ANALYSIS

As described, the proposed measurement scheme involves the determinations of two parameters: the measurement strength $\theta$ and the purity $P_m$. These requirements are not specific to our scheme. In all weak-value-measurement protocols, the determination of the measurement strength $\theta$ is a necessary and inevitable process which requires a separate acquisition. Our scheme does not perform better or worse than other schemes in the literature in that respect. The additional determination of the purity $P_m$ is only required because we considered the most general case of an initial incoherent state of the qubit meter system. Most of the literature assumes the meter to be in a known pure state to avoid this supplementary step. To determine the initial meter state in our protocol, it is only necessary to perform the quantum tomography of a single qubit. In practice, in our experimental implementation, we simulated the CNOT gate by using spontaneous parametric down conversion (instead of using a true CNOT gate with two separate entries that could be characterized independently). For this reason, in our experiment, we determined the purity by performing quantum tomography of the two-qubit state since the input meter state could not be measured directly. This two-qubit tomography is not at all required by the proposed protocol, but appears only as a side effect of the practical implementation of our demonstrative experiment.

[26] A simple transformation of $\hat{U} = e^{-i(\hat{H}_{int}+\hat{H})t}$ with the interaction Hamiltonian $\hat{H}_{int} = g(t)\hat{\Omega}_{\alpha} \otimes \hat{\Lambda}$ reveals the similitude between the von Neumann measurement scheme and our gate application $\hat{U}_{\text{GATE}}$. 